Approval: 23rd Senate Meeting

Course Name	: Operator Theory
Course Number	: MA-612
Credit	: 3-1-0-4
Prerequisites	: MA 521 (Functional Analysis)
Students intended for	: M.Sc./B.Tech/M.S./M.Tech/Ph.D.
Elective or core	: Elective
Semester	: Odd/Even

Preamble: The objective of this course is to introduce fundamental topics in operator theory. It is a field that has great importance for other areas of mathematics and physics, such as algebraic topology, differential geometry, and quantum mechanics. It assumes a basic knowledge in functional analysis but no prior acquaintance with operator theory is required.

In the first part of this course, we discuss the basic results of spectral theory. The most important of these are the non-emptiness of the spectrum, Beurling's spectral radius formula, and the Gelfand representation theory for commutative Banach algebras. We discuss compact and Fredholm operators and describes their elementary theory. Important concepts are the essential spectrum and the Fredholm index. The ground field for all vector spaces and algebras is the complex field **C**. A complete normed algebra is called a Banach algebra. A complete unital normed algebra is called a unital Banach algebra. One thinks of the spectrum as simultaneously a generalization of the range of a function and the set of eigenvalues of a finite square matrix. According to Gelfand theorem, if **a** is an element of a unital Banach algebra A, then the spectrum σ(**a**) of **a** is nonempty. There are algebras in which not all elements have nonempty spectrum.

In the second part, we present a study of C*-algebras and of operators on Hilbert spaces. Hilbert spaces are very well-behaved compared with general Banach spaces, and the same is true of C*-algebras as compared with general Banach algebras. Here the main results of are a theorem of Gelfand, which asserts that up to isomorphism all abelian C*-algebras are of the form $C_0(\Omega)$, where Ω is a locally compact Hausdorff space, and the spectral theorem. The spectral theorem enables to synthesize a normal operator from linear combinations of projections where the coefficients lie in the spectrum. We also discuss a partial order relation on the Hermitian elements of a C*-algebra. The principal results are the existence of a unique positive square root for each positive element and the theorem, which asserts that elements of the form **a*a** are positive.

In the end, we prove that every C*-algebra can be realized as a C*-subalgebra of B (H) for some Hilbert space H. This is the Gelfand–Naimark theorem, and it is one of the fundamental results of the theory of C*-algebras. A key step in its proof is the GNS construction that sets up a correspondence between the positive linear functionals and some of the representations of the algebra. There are also deep connections between the positive linear functionals and the closed ideals and closed left ideals of the algebra.

Module 1: Elementary Spectral Theory [18 Hours]

Banach Algebras, Examples of Banach Algebra, Spectrum and the Spectral Radius, Neumann Series, The Fundamental Theorem of Banach Algebra by Gelfand, Gelfand-Mazur Theorem, The Beurling Theorem for Spectral Radius, The Gelfand Representation, Compact and Fredholm Operators, Integral Operators, Kernels of the Integral Operators, Volterra Integral Operator, Transpose of the Bounded Linear Maps between Banach Spaces, Bounded Below Linear Maps between Banach Spaces, Ascent and Descent, Index of Bounded Linear Maps between Banach Spaces, The Fundamental Result of Fredholm Theory, Fredholm Alternative, Characterization of Fredholm Operators, Essential Spectrum in terms of Fredholm.

Module 2: C*-Algebras and Hilbert Space Operators [18 Hours]

Involution on an Algebra, *-Algebra, *-Algebra Generated by a Subset of *-Algebra, *-homomorphism between *-Algberas, Banach *-Algebras, Unital Banach *-Algebras, C*-Algebras, Examples of C*-Algberas, Double Centralizer for a C*-Algebra, Multiplier Algebra, Complete Characterization of Abelian C*-Algebras using the Gelfand Representation, Spectral Mapping Theorem, Positive Elements of C*-Algebras, Operators and Sesquilinear Forms, Adjoint and its properties, Orthogonal Projections, Invariant Subspaces, Reducing Subspaces, Partial Isometries and their Characterizations, Polar Decomposition, Compact Hilbert Space Operators, Diagonalizable Operators, Diagonalizability of Compact Normal Operators on Hilbert Spaces, denseness of the set of finite-rank operators, Hilbert-Schmidt Operators, Trace-Class Operators, The Spectral Theorem for Normal Operators.

Module 3: Gelfand–Naimark Theorem [6 Hours]
Ideals in C*-algebras, Positive Linear Functionals, Gelfand-Naimark-Segal Represenatation.

Text Books:

1. Gerard J. Murphy, *C*-algebras and Operator Theory*, Academic Press, 2014.

2. Ronald G. Douglas, *Banach Algebra Techniques in Operator Theory*, Volume 179 of Graduate Texts in Mathematics, Springer, 2012.

Reference Books:

1. **Kenneth R. Davidson, *C*-algebras by Example*, American Mathematical Soc., 1996.**

2. **Kehe Zhu, *An Introduction to Operator Algebras*, Volume 9 of Studies in Advanced Mathematics, CRC Press, 2018.**

3. **John B. Conway, *A Course in Operator Theory*, American Mathematical Soc., 2000.**

Similarity content declaration with existing courses:

Sr. No.	**Course**	**Similarity Content**	**Approximate % of Content**
1.	MA 521	Compact Operators	2 %

Justification for new course proposal if cumulative similarity content is > 30%: Not Applicable